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ULTRASONIC SCANNING OF AN ACOUSTIC MEDIUM THROUGH A TWO-LAYER PLATE AT THE CRITICAL FREQUENCY†

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The plane problem of ultrasonic scanning of an acoustic medium through a two-layer plate at high frequencies is investigated. In passing, the problem of the localization of oscillations in the neighbourhood of the finite region where the load is applied in such a way as to eliminate the propagation of sonic energy along the plate is solved. The possibility of radiation patterns of acoustic pressure being formed in the acoustic medium is investigated by the stationary-phase method. The theory constructed is illustrated by a series of calculations for a steel–rubber–water system.

The effect of the scanning of an acoustic medium through a solid wall (an elastic uniform or laminated plate) depends on a number of factors. For example, if oscillations of a plate-liquid system are excited by a harmonically varying normal pressure, distributed over a finite area S of the plate surface, not in contact with the liquid, then, since the plate is a waveguide, for high-frequency oscillations it is extremely probable that a considerable part of the energy supplied will propagate along the plate. In this case, even in ideal systems (ignoring internal losses in the plate material and in the liquid) scanning will not be effective. Hence, the problem arises of localizing the oscillations in the neighbourhood of the region where the external pressure is applied. This problem has been investigated in some detail for a uniform plate without a liquid [1] and for a uniform plate lying on the surface of a liquid [2], where the frequency of excitation of the oscillations is identical with one of the critical frequencies of the plate (the cutoff frequency). The effect of scanning is also determined by the radiation pattern of the sonic field in the liquid, which, for specified parameters of the plate, depends considerably on the amplitude distribution of the pressure and the geometrical parameters of the region S.

In this paper we consider the problem of the propagation and decay of waves in an elastic two-layer strip, lying on the surface of an ideal compressible liquid, we solve the problem of the localization of the oscillations and we investigate the radiation pattern of the acoustic pressure in the far field.

1. FORMULATION OF THE PROBLEM AND CONSTRUCTION OF THE SOLUTION

We will consider the problem of the propagation of waves in a two-layer elastic strip, lying on the surface of an ideal compressible liquid.

We choose a Cartesian system of coordinates x_1, x_2 such that $S_1 = \{-\infty < x_1 < \infty, 0 \le x_2 \le h_1\}$ and $S_2 = \{-\infty < x_1 < \infty, h_1 \le x_2 \le h\}$ are subregions of the composite strip, h_1 and h_2 are the thicknesses of the layers and $h = h_1 + h_2$. $S_0 = \{-\infty < x_1 < \infty, h \le x_2 \le \infty\}$ is the region occupied by the liquid. We will denote by μ_j, ν_j, ρ_j the shear modulus, Poisson's ratio and the density of the materials (j = 1, 2, respectively); ρ_0 is the density of the liquid and c_0 is the velocity of sound in the liquid.

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Suppose the wave processes in the regions $S_1 \cup S_2 \cup S_0$ are excited by normal stresses applied to the faces $x_2 = 0$, and vary harmonically as $\exp(-i\omega t)$. The problem reduces to finding solutions in the regions S_j of the equations of harmonic oscillations

$$\begin{aligned} \chi_j\partial_1^2 u_{j1} + (\chi_j - 2\mu_j)\partial_1\partial_2 u_{j2} + \mu_j(\partial_2^2 u_{j2} + \partial_1\partial_2 u_{j2}) + \rho_j\omega^2 u_{j1} &= 0 \\ \chi_j\partial_1^2 u_{j2} + (\chi_j - 2\mu_j)\partial_1\partial_2 u_{j1} + \mu_j(\partial_2^2 u_{j1} + \partial_1\partial_2 u_{j2}) + \rho_j\omega^2 u_{j2} &= 0 \end{aligned} \tag{1.1}$$

where $\partial_j = \partial/\partial x_j$, u_j are the projections of the amplitudes of the displacement oscillations onto the x_j axes respectively, $\chi_j = 2\mu_j\nu_j/(1 - 2\nu_j)$ and, in the region S_0, of the wave equation

$$\Delta\varphi + k_0^2\varphi = 0, \quad \Delta = \partial_1^2 + \partial_2^2 \tag{1.2}$$

for the following boundary conditions at the interfaces of the media

$$x_2: \sigma_{j12} = 0, \quad \sigma_{j12} = \mu_1 g(x_1) \tag{1.3}$$

$$x_2 = h_1: u_{1\beta} - u_{2\beta} = 0, \quad \sigma_{11\beta} - \sigma_{21\beta} = 0, \quad \beta = 1, 2 \tag{1.4}$$

$$x_2 = h: -i\omega u_{22} - \partial_2\varphi = 0, \quad \sigma_{222} + p = 0, \quad \sigma_{212} = 0 \tag{1.5}$$

Here

$$\sigma_{j12} = \mu_j(\partial_1 u_{j2} + \partial_2 u_{j1}), \quad \sigma_{j22} = \chi_j\partial_2 u_{j2} + (\chi_j - 2\mu_j)\partial_1 u_{j1}$$

$$p = -i\rho_0\omega\varphi$$

are the amplitudes of the stresses and pressure in the liquid, respectively, and φ is the amplitude of the velocity potential at points of the liquid medium.

To construct the solution of the problem we will change to dimensionless coordinates $\xi = x_1/h$, $\zeta = x_2/h$ and convert relations (1.1)–(1.5) using a Fourier transformation with respect to the variable ξ. We introduce the four-component vector

$$\mathbf{Y}_j = [y_{j1}, y_{j2}, y_{j3}, y_{j4}]$$

$$y_{j1} = u_{j1}^*/h, \quad y_{j2} = -iu_{j2}^*/h, \quad y_{j3} = \sigma_{j12}^*/\mu_1, \quad y_{j4} = -i\sigma_{j22}^*/\mu_1$$

where

$$f^* = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)\exp(i\gamma\xi)d\xi$$

In terms of Fourier transforms, taking into account the notation employed, we write the system of equations (1.1), the solution of Eq. (1.2), the boundary conditions (1.3) and the conditions of continuity (1.4) and (1.5) in the form

$$\mathbf{Y}_j' = \mathbf{A}_j\mathbf{Y}_j \tag{1.6}$$

$$\varphi^*(\zeta) = \varphi_0\exp[iq_0(\zeta - 1)], \quad q_0 = \sqrt{\frac{\Omega^2}{\theta^2} - \gamma^2} \tag{1.7}$$

$$y_{13}(0) = 0, \quad y_{14}(0) = -ig^* \tag{1.8}$$

$$\mathbf{Y}_1(\zeta_1) = \mathbf{Y}_2(\zeta_1) \tag{1.9}$$

$$y_{j1}(1) = -\varepsilon_1\Omega^{-1}q_0\psi_0, \quad y_{j4}(1) = i\varepsilon_2\Omega\psi_0; \quad \psi_0 = \frac{\varphi_0}{hc_0} \tag{1.10}$$

where

$$\mathbf{A}_j = \left\| \begin{array}{cccc} 0 & \gamma & \dfrac{1}{\mu_j^0} & 0 \\ \gamma^2(k_j^{-2}-1) & 0 & 0 & \dfrac{1}{p_{j1}\mu_j^0} \\ \mu_j^0[4\gamma^2(1-k_j^{-2}-\Omega_j^2] & 0 & 0 & \gamma(1-2k_j^{-2}) \\ 0 & -\mu_j^0\Omega_j^2 & -\gamma & 0 \end{array} \right\|$$

$$k_j^2 = 2\frac{1-\nu_j}{1-2\nu_j}, \quad \Omega_j^2 = \frac{\rho_j\omega^2h^2}{\mu_j}, \quad \varepsilon_1 = \frac{\rho_0}{\rho_1}, \quad \varepsilon_2 = \frac{c_0\rho_0}{c_{1t}\rho_1}, \quad c_{jt} = \sqrt{\frac{\mu_j}{\rho_j}}, \quad \mu_j^0 = \frac{\mu_j}{\mu_1}$$

The prime denotes a derivative with respect to ζ.

Here $\Omega = \Omega_1$ is the fundamental dimensionless frequency parameter and c_{jt} is the velocity of transverse waves in the material of the j-component.

We will construct an evolution operator of the system of differential equations (1.6), taking into account the interface condition (1.9). To do this we choose an arbitrary system of four linearly independent vectors of Eq. (1.6). We can take as such vectors

$$\begin{aligned} \mathbf{Y}_{j1} &= [\gamma\,\mathrm{ch}(p_{j1}\zeta), -p_{j1}\mathrm{sh}(p_{j1}\zeta), 2\mu_j^0\gamma p_{j1}\mathrm{sh}(p_{j1}\zeta), \mu_j^0(2\gamma^2-\Omega_j^2)\mathrm{ch}(p_{j1}\zeta)] \\ \mathbf{Y}_{j2} &= [\gamma\,\mathrm{sh}(p_{j1}\zeta), -p_{j1}\mathrm{ch}(p_{j1}\zeta), 2\mu_j^0\gamma p_{j1}\mathrm{ch}(p_{j1}\zeta), \mu_j^0(2\gamma^2-\Omega_j^2)\mathrm{sh}(p_{j1}\zeta)] \\ \mathbf{Y}_{j3} &= [-p_{j2}\mathrm{ch}(p_{j2}\zeta), \gamma\,\mathrm{sh}(p_{j1}\zeta), -\mu_j^0(2\gamma^2-\Omega_j^2)\mathrm{sh}(p_{j1}\zeta), 2\mu_j^0\gamma p_{j2}\mathrm{ch}(p_{j2}\zeta)] \\ \mathbf{Y}_{j4} &= [-p_{j2}\mathrm{sh}(p_{j2}\zeta), \gamma\,\mathrm{ch}(p_{j1}\zeta), -\mu_j^0(2\gamma^2-\Omega_j^2)\mathrm{ch}(p_{j1}\zeta), 2\mu_j^0\gamma p_{j2}\mathrm{sh}(p_{j2}\zeta)] \\ p_{j1} &= \sqrt{\gamma^2-\frac{\Omega_j^2}{k_j^2}}, \quad p_{j2} = \sqrt{\gamma^2-\Omega_j^2} \end{aligned}$$

We introduce the matrix operator functions $\mathbf{B}_j(\zeta)$, the columns of which are the vectors $\mathbf{Y}_{jm}$ (m = 1, 2, 3, 4). We have

$$\mathbf{B}_j(\zeta) = [\mathbf{Y}_{j1}, \mathbf{Y}_{j2}, \mathbf{Y}_{j3}, \mathbf{Y}_{j4}]$$

We introduce the notation $\mathbf{B}_{j0} = \mathbf{B}_j(0)$, $\mathbf{V}_j = \mathbf{B}_{j0}^{-1}$ and construct the operator functions

$$\mathbf{U}_j(\zeta) = \mathbf{B}_{j0}(\zeta)\mathbf{V}_j$$

Obviously $\mathbf{U}_j(0) = \mathbf{I}$, where $\mathbf{I}$ is the identity operator.

We define the evolution operator as follows:

$$\mathbf{U}(\zeta) = \begin{cases} \mathbf{U}_1(\zeta) & \text{for} \quad 0 \le \zeta \le \zeta_1 \\ \mathbf{U}_2(\zeta-\zeta_1)\mathbf{U}_1(\zeta_1) & \text{for} \quad \zeta_1 \le \zeta \le 1 \end{cases} \tag{1.11}$$

The operator $\mathbf{U}(\zeta)$ makes the vector $\mathbf{Y}(\zeta)$ correspond to each vector $\mathbf{Y}_0 = [y_{10}, y_{20}, y_{30}, y_{40}]$, where $y_{m0} = y_m(0)$; the vector $\mathbf{Y}(\zeta)$ satisfies Eqs (1.6) and the continuity condition (1.9).

We will denote the elements of the matrices $\mathbf{U}(\zeta)$ and $\mathbf{U} = \mathbf{U}(1)$ by $U_{mn}(\zeta)$ and U_{mn} (m, n = 1, 2, 3, 4) respectively. Taking the notation employed and boundary conditions (1.8) into account we have

$$\mathbf{Y}_0 = [y_{10}, y_{20}, 0, g^*]$$

where y_{10} and y_{20} are arbitrary constants, while the interface conditions (1.10) take the form

$$
\begin{aligned}
U_{21}y_{10} + U_{22}y_{20} &= -U_{24}g^* - \varepsilon_1\Omega^{-1}q_0\psi_0 \\
U_{31}y_{10} + U_{32}y_{20} &= -U_{31}g^* \\
U_{41}y_{10} + U_{42}y_{20} &= -U_{44}g^* + i\mu_1\varepsilon_2\Omega\psi_0
\end{aligned}
\tag{1.12}
$$

Solving system of equations (1.12) for y_{10}, y_{20}, ψ_0, we obtain

$$
y_{\beta 0} = F^{-1}z_\beta g^*, \quad z_\beta = q_0 f_{\beta 0} + i\varepsilon\Omega^2 f_{\beta 1}, \quad \beta = 1, 2
$$

$$
\psi_0 = \varepsilon_1^{-1}F^{-1}\Omega f g^*, \quad F = q_0F^0 - i\varepsilon\Omega^2F^1, \quad \varepsilon = \frac{c_0}{c_{1t}}
$$

$$
\begin{aligned}
F^0 &= U_{31}U_{42} - U_{32}U_{41}, \quad F^1 = U_{31}U_{22} - U_{32}U_{21} \\
f_{10} &= U_{32}U_{44} - U_{34}U_{42}, \quad f_{11} = U_{32}U_{24} - U_{22}U_{34} \\
f_{20} &= U_{34}U_{41} - U_{31}U_{44}, \quad f_{21} = U_{21}U_{34} - U_{31}U_{24} \\
f &= U_{34}(U_{21}U_{42} - U_{22}U_{41}) - U_{44}(U_{21}U_{32} - U_{22}U_{31})
\end{aligned}
\tag{1.13}
$$

The components of the displacement vector of the points of the plate and the pressure in the liquid can be represented in the form of the following integrals

$$
u_m(\xi, \zeta) = (i)^{m-1}\frac{h}{2\pi}\int_{-\infty}^{\infty}\frac{U_m(\zeta, \gamma)}{F(\gamma)}g^*\exp(i\gamma\xi)d\gamma, \quad m = 1, 2 \tag{1.14}
$$

$$
p(\xi, \zeta) = \frac{p_0}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\zeta, \gamma)}{F(\gamma)}g^*\exp[iq_0(\zeta - 1)]\exp(i\gamma\xi)d\gamma \tag{1.15}
$$

$$
p_0 = \varepsilon\mu_1\Omega^2, \quad U_m(\zeta, \gamma) = z_1U_{m1}(\zeta) + z_2U_{m2}(\zeta) + U_{m3}(\zeta)
$$

2. CRITICAL FREQUENCIES AND MODES OF A TWO-LAYER STRIP

A general theory of the definition of critical frequencies and the modes corresponding to them in the case when the elastic characteristics of the plate are arbitrary piecewise-continuous functions of the transverse coordinate has been given in [3, 4]. The results described in these papers are obtained in a somewhat different form below for a two-layer strip, the faces of which are stress-free.

Thus, in the region $S^0 = S_1 \cup S_2$ we consider Eqs (1.1) with the homogeneous boundary conditions

$$
x_2 = 0\colon \sigma_{j12} = 0, \quad \sigma_{j12} = 0; \quad x_2 = h\colon \sigma_{j12} = 0, \quad \sigma_{j12} = 0 \tag{2.1}
$$

and the continuity conditions (1.4).

After changing to dimensionless coordinates and separating the variables, we obtain the following two-parameter eigenvalue problem

$$
\begin{aligned}
\mathbf{Y}_j' &= \mathbf{A}_j(\gamma, \Omega)\mathbf{Y}_j \\
y_{13}(0) = 0, \quad y_{14}(0) &= 0, \quad y_{13}(1) = 0, \quad y_{14}(1) = 0
\end{aligned}
\tag{2.2}
$$

Using the evolution operator (1.11), to construct non-trivial solutions we obtain the dispersion equation

$$
F^0(\gamma, \Omega) = 0 \tag{2.3}
$$

We will call the quantities γ and Ω for which Eq. (2.3) is satisfied a spectral pair (γ, Ω). Those values of $\Omega = \Omega_c$ for which Eq. (2.3) has multiple roots γ_c will be called critical values. We will call the spectral pair (γ_c, Ω_c) a critical pair.

It is well known [5, 6], that $\gamma_c = 0$ is a multiple eigenvalue. As a rule, the multiplicity is equal to 2, but for certain values of the parameters the multiplicity may be equal to 4 [4, 7].

Substituting $\gamma = 0$ into Eq. (2.3), we obtain

$$F^0(0, \Omega) = D_1(\Omega)D_2(\Omega) = 0 \tag{2.4}$$

where

$$D_1(\Omega) = \cos(\zeta_1\Omega)\sin(m_t\zeta_2\Omega) + \kappa_t\sin(\zeta_1\Omega)\cos(m_t\zeta_2\Omega)$$

$$D_2(\Omega) = \cos(m_{1l}\zeta_1\Omega)\sin(m_{2l}\zeta_2\Omega) + \kappa_l\sin(m_{1l}\zeta_1\Omega)\cos(m_{2l}\zeta_2\Omega)$$

$$c_{jl} = \sqrt{\frac{\chi_j}{\rho_j}}, \quad \kappa_t = \frac{c_{1t}\rho_1}{c_{2t}\rho_2}, \quad \kappa_l = \frac{c_{1l}\rho_1}{c_{2l}\rho_2}, \quad m_t = \frac{c_{1t}}{c_{2t}}, \quad m_{1l} = \frac{c_{1t}}{c_{1l}}, \quad m_{2l} = \frac{c_{2t}}{c_{2l}}$$

The parameters κ_t and κ_l represent the ratio of the wave impedances of the transverse and longitudinal waves respectively.

The zeros of the functions $D_1(\Omega)$ and $D_2(\Omega)$ will be called critical frequencies of the first and second kinds respectively and we will denote them by Ω_{1r} and Ω_{2r} $(r = 1, 2, \ldots)$.

We will denote the eigenvector corresponding to the critical spectral pair by $\mathbf{Y}^0 = \{\mathbf{Y}_1^0, \mathbf{Y}_2^0\}$, $\mathbf{Y}_j^0 = [y_{j1}^0, y_{j2}^0, y_{j3}^0, y_{j4}^0]$. Its j-components have the following structure: $\mathbf{Y}_j^0 = [y_{j1}^0, 0, 0, y_{j4}^0]$ for Ω_{1r} and $\mathbf{Y}_j^0 = [0, y_{j2}^0, y_{j3}^0, 0]$ for Ω_{2r}. Since $\gamma_c = 0$ is a multiple eigenvalue, it is possible for an associated vector $\mathbf{Y}^1 = \{\mathbf{Y}_1^1, \mathbf{Y}_2^1\}$ to exist, the j-components of which are defined by the solution of the inhomogeneous problem

$$\begin{gathered}\mathbf{Y}_j^{1\prime} = \mathbf{A}_j(0, \Omega_c)\mathbf{Y}_j^1 + \mathbf{Y}_j^0 \\ y_{j3}^1(0) = 0, \quad y_{j4}^1(0) = 0, \quad y_{j3}^1(1) = 0, \quad y_{j4}^1(1) = 0\end{gathered} \tag{2.5}$$

and have the following structure: $\mathbf{Y}_j^1 = [0, y_{j2}^1, y_{j3}^1, 0]$ for Ω_{1r} and $\mathbf{Y}_j^1 = [y_{j1}^1, 0, 0, y_{j4}^1]$ for Ω_{2r}.

If the integral

$$d = \int_0^1 \mathbf{Y}^1 \cdot \mathbf{J}\mathbf{Y}^0 d\zeta \neq 0 \tag{2.6}$$

where

$$\mathbf{J} = \begin{Vmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{Vmatrix}$$

(**0** is the null matrix and **I** is the 2×2 identity matrix), no other associated vectors exist. In this case the initial problem (1.1), (2.1) has two elementary solutions:

When $\Omega = \Omega_{1r}$

$$\mathbf{w}_j^0 = [y_{j1}^0, 0, 0, y_{j4}^0], \quad \mathbf{w}_j^1 = [i\xi y_{j1}^0, iy_{j2}^1, iy_{j3}^1, -\xi y_{j4}^0] \tag{2.7}$$

When $\Omega = \Omega_{2r}$

$$\mathbf{w}_j^0 = [0, y_{j2}^0, y_{j3}^0, 0], \quad \mathbf{w}_j^1 = [y_{j1}^1, -\xi y_{j2}^0, i\xi y_{j3}^0, iy_{j4}^1] \tag{2.8}$$

The elementary solution $\mathbf{w}_j^0$ describe thickness resonances (longitudinal and transverse). The elementary solutions $\mathbf{w}_j^1$, as follows from expressions (2.7) and (2.8), increase without limit as $\xi \to \pm\infty$, which is a consequence of the assumption that the materials of the strips have ideal elasticity and there is no radiation of energy through the faces by virtue of the assumed boundary conditions (2.1).

Note that condition (2.6) is equivalent to the condition

$$F^0_{,\gamma\gamma}(0, \Omega_c) \neq 0 \tag{2.9}$$

If $d = 0$, there are at least two other associated vectors (2.3) $\mathbf{Y}^s = \{\mathbf{Y}^s_1, \mathbf{Y}^s_2\}$ ($s = 2, 3$), the j-components of which are defined by the solution of the non-homogeneous problems

$$\mathbf{Y}^{s\prime}_j = \mathbf{A}_j(0, \Omega_c)\mathbf{Y}^s_j + \mathbf{Y}^{s-1}_j$$

$$y^s_{j3}(0) = 0, \quad y^s_{j4}(0) = 0, \quad y^s_{j3}(1) = 0, \quad y^s_{j4}(1) = 0$$

In this case there are two other pairs of elementary solutions with a higher-power increase [4, 7].

Note that the resonance modes do not transfer energy along the strip, since their specific energy flux through the transverse cross section of the strip, averaged over a period, is equal to

$$P^0(\mathbf{w}^0) = \frac{ih\mu_1\omega}{2}\int_0^1 \mathbf{w}^0 \cdot \mathbf{J}\mathbf{w}^0 d\zeta = 0, \quad \mathbf{w}^0 = \{\mathbf{w}^0_1, \mathbf{w}^0_2\}$$

This property of resonance modes, as will be seen later, turns out to be important in the problem of localizing the oscillations in the neighbourhood of the finite region in which the external load is applied $\sigma_{122}|_{\zeta=0} = \mu_1 g(\xi)$.

To conclude this section we note that, for each resonance frequency, Eq. (2.3) has finite number of $2N$ real roots $\alpha^\pm_n = \pm\alpha_n$ ($n = 1, \dots, N$; $\alpha_n > 0$) and a denumerable symmetrical set of complex roots γ_k. It is well known [7, 8], that the energy fluxes of modes corresponding to the real roots (uniform modes) are non-zero, while the energy fluxes corresponding to the complex roots (non-uniform modes) equal zero. Hence, only uniform modes are "responsible" for the transfer of energy in an unbounded waveguide.

3. QUASICRITICAL AND QUASIUNIFORM MODES AND THE LOCALIZATION OF OSCILLATIONS

We now return to the initial problem. We will assume that the amplitude of the pressure $\mu_1 g(\xi)$, specified when $\zeta = 0$, is non-zero in the region $-a \le \xi \le a$. Before analysing the integral representation of solution (1.14), we will first investigate the equation

$$F(\gamma, \Omega) = q_0 F^0 - i\varepsilon\Omega^2 F^1 = 0 \tag{3.1}$$

on the assumption that the parameter $\varepsilon \ll 1$. In this case the distribution of the roots of this equation, knowing the distribution of the roots of Eq. (2.3), can be determined by methods of perturbation theory.

Suppose $\Omega = \Omega_c$. In this case Eq. (3.1) has multiple wave numbers $\gamma_0 = 0$ only when $\Omega_c = \Omega_{1r}$, since an ideal liquid has no effect on the longitudinal oscillations of the strip, which is also confirmed by an analysis of Eq. (3.1). When $\Omega_c = \Omega_{2r}$ the multiplicity is lost, and for small ε a pair of complex wave numbers appears (in the case of a multiplicity of 2) of the form

$$\begin{aligned} \gamma^\pm_0 &= \pm\varepsilon^{1/2}|\alpha_0|^{1/2}(1 + i\,\mathrm{sign}\,\alpha_0) + O(\varepsilon^{3/2}) \\ \alpha_0 &= -q_0\Omega^2_c F^1(0, \Omega_c)/F^0_{,\gamma\gamma}(0, \Omega_c) \end{aligned} \tag{3.2}$$

These wave numbers will be called quasi-critical wave numbers, and the corresponding modes will be called quasi-critical modes. The approximate numerical values, determined from formula (3.2), can easily be refined by numerical methods. It follows from formula (3.2) that quasi-critical modes of this kind are decaying modes with a decay coefficient proportional to $\varepsilon^{1/2}$.

Unlike quasi-critical modes, quasi-uniform modes, which occur as a result of perturbation of uniform modes, decay much more slowly. The wave numbers corresponding to them for small ε are given by the following analytical expressions

$$\gamma_n^{\pm} = \pm\alpha_n \mp i\varepsilon\Omega_c^2 F^1(0, \Omega_c)/F^0_{,\gamma}(0, \Omega_c) + O(\varepsilon^2) \tag{3.3}$$

It can be seen from (3.2) and (3.3) that, of the pair of quasi-critical wave numbers, one has a positive imaginary part, while the second has a negative imaginary part; the set $\gamma_n^{\mp}$ has similar properties. Henceforth γ_n^+ ($n = 0, 1, \ldots, N$) denotes that $\mathrm{Im}\,\gamma_n^+ > 0$. Similar notation is also used for the remaining roots of Eq. (3.1), which tend to the complex roots of Eq. (2.3) as $\varepsilon \to 0$.

Suppose $\Omega = \Omega_{2r}$. Consider an arbitrary field characteristic, for example, u_2. We will transform the integral on the right-hand side of Eq. (1.14) using the theory of residues. Taking into account the fact that the integrand has a branching point $\gamma_*^{\pm} = \pm\Omega_2/\theta$ for $\xi \geq a$, we have

$$u_2 = \sum_{n=0}^{N} \frac{U_2(\zeta, \gamma_n^+)}{F_{,\gamma}(\gamma_n^+)} C_n' \exp(i\gamma_n^+\xi) + \sum_k \frac{U_2(\zeta, \gamma_k^+)}{F_{,\gamma}(\gamma_k^+)} C_k' \exp(i\gamma_k^+\xi) + \int_{\Gamma^+} \frac{U_2(\zeta, \gamma)}{F(\gamma)} g^* \exp(i\gamma\xi) d\gamma \tag{3.4}$$

$$C_p' = \int_{-a}^{a} g(\xi)\exp(-i\gamma_n^+\xi)d\xi, \quad p = n, k$$

where Γ^* is a contour in the form of a loop departing to infinity, situated in the right half-plane with vertex at the point $\gamma = \gamma_*^+$ [8]. Since, as $\varepsilon \to 0$ the integral along the loop vanishes, then, as follows from formula (3.4), to localize the oscillations in the neighbourhood of the application of the load, it is sufficient to choose the function $g(\xi)$ such that $C_n' = 0$ ($n = 1, \ldots, N$), i.e. in other words, the quasi-uniform modes are "cut-off". Obviously such a problem has a non-unique solution and allows off a certain freedom of choice of the function $g(\xi)$ for controlling the directional characteristics of the sound field in the liquid.

In this investigation we sought the load in the form

$$g(\xi) = g_0\left(1 + \sum_{i=1}^{N} l_i \xi^{2i}\right) \tag{3.5}$$

The coefficients l_i were determined by solving the algebraic system which arises from the conditions

$$C_n = \int_{-a}^{a} \cos(\alpha_n\xi)g(\xi)d\xi = 0, \quad n = 1, \ldots, N \tag{3.6}$$

The replacement of the conditions $C_n' = 0$ ($n = 1, \ldots, N$) by the conditions (3.6) for small ε has little effect on the final results, but leads to an algebraic system with real coefficients.

4. THE DIRECTIVITY CHARACTERISTIC

To construct the directivity characteristic of the pressure field in the far-field region we turn to formula (1.15).

Using the replacement

$$\xi = \rho\cos\tau, \quad \zeta = 1 + \rho\sin\tau, \quad \tau \in (0, \pi), \quad \rho \in (0, \infty)$$

we convert the integral to the form

$$p(\rho, \tau) = \int_{-\infty}^{\infty} U(\gamma)\exp[i\rho K(\gamma, \tau)]d\gamma \tag{4.1}$$

where

$$U(\gamma) = \frac{p_0}{2\pi i}\frac{f(\gamma)}{F(\gamma)}g^*, \quad K(\gamma, \tau) = \gamma\cos\tau - \sqrt{\gamma_*^2 - \gamma^2}\sin\tau$$

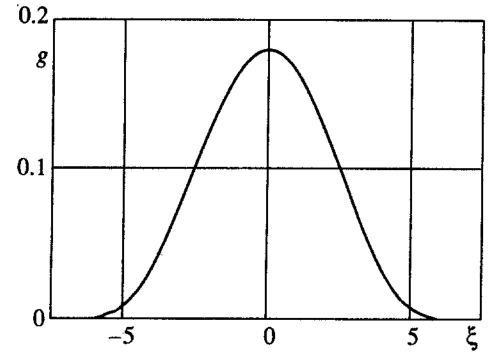

Fig. 1

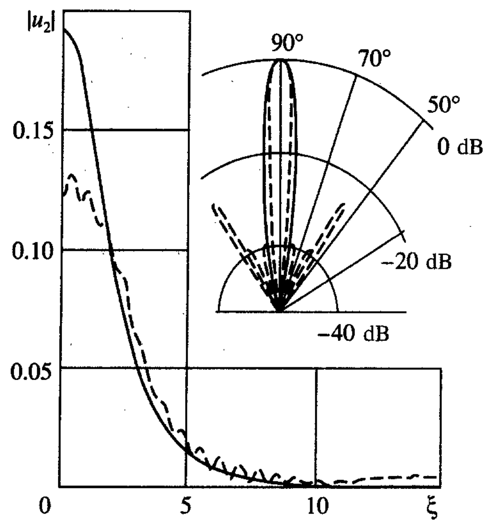

Fig. 2

Using the stationary-phase method [9], we determine the point γ_{st}, at which the phase takes a stationary value. We have

$$\gamma_{st} = -\Omega_{2r}|\cos\tau|/\theta \tag{4.2}$$

Converting expression (4.1), taking expression (4.2) into account, we obtain an asymptotic formula for calculating the sound pressure

$$p(\rho, \tau) = \sqrt{\frac{2\pi\Omega_{2r}}{\rho\theta}}U\left(-\frac{\Omega_{2r}|\cos\tau|}{\theta}\right)\sin\tau\exp\left(-\frac{i\rho\Omega_{2r}}{\theta}+\frac{i\pi}{4}\right)+O(\rho^{-1}) \tag{4.3}$$

We also obtain a formula for constructing the directivity characteristic

$$d(\tau) = 8.6859\ln\left|\frac{p(\rho,\tau)}{p(\rho,\pi/2)}\right| = 8.6859\ln\left|\frac{U(-\Omega_{2r}|\cos\tau|)/\theta}{U(0)}\sin\tau\right| \tag{4.4}$$

5. THE RESULTS OF CALCULATIONS

We will present some results of a numerical investigation of the problem. Calculations were carried out for a steel-rubber-water system with the following values of the parameters

$$h = 4 \cdot 10^{-3}\ \text{m},\quad h_1 = h_2 = 2 \cdot 10^{-3}\ \text{m},\quad \mu_1 = 7.84 \cdot 10^{10}\ \text{N/m}^2,\quad \mu_2 = 3.92 \cdot 10^{9}\ \text{N/m}^2,$$

$$\nu_1 = 0.29,\quad \nu_2 = 0.45,\quad \rho_1 = 7.18 \cdot 10^3\ \text{kg/m}^3,\quad \rho_2 = 2 \cdot 10^3\ \text{kg/m}^3,$$

$$\rho_0 = 10^3\ \text{kg/m}^3,\quad c_0 = 1490\ \text{m/s},\ a = 6$$

and for the first critical frequency $\Omega_{21} = 2.3845$. Then the wave numbers of the quasi-critical modes are

$$\gamma_c = \mp 0.1794 \pm 0.1823i$$

the wave numbers of the uniform modes when there is no liquid are

$$\alpha_1 = 2.631,\quad \alpha_2 = 2.903,\quad \alpha_3 = 5.546$$

and the wave numbers of the quasi-uniform modes are

$$\gamma_1^{\pm} = \mp 2.633 \pm 0.0403i,\quad \gamma_2^{\pm} = \mp 2.902 \pm 0.0106i,\quad \gamma_3^{\pm} = \mp 5.547 \pm 0.0298i$$

The imaginary parts $\gamma_c^{\pm}$, $\gamma_n^{\pm}$ clearly illustrate the difference in the rates of decay of the quasi-critical and quasi-uniform modes. Note that, in this problem, the decay of the chosen modes is due to radiation of sonic energy into the semi-bounded acoustic medium (into the water).

In Fig. 1 we show a graph of the distribution of the amplitude of the external pressure $g(\xi)$ (we have used the normalization $\int_{-a}^{a} g(\xi)d\xi = 1$), for which the amplitudes of the uniform modes in the regions $|\xi| \geq a$ are equal to zero.

In Fig. 2 we show graphs of $|u_{22}(\xi, 1)|$; the continuous curve corresponds to the pressure distribution $g(\xi)$ and illustrates the localization of the oscillations in the neighbourhood of the region where the external load is applied, while the dashed curve represents the pressure distribution $g^0(\xi) = 0.08333$ ($|\xi| \leq a$).

In the right upper part of Fig. 2 we show the directivity characteristic of the pressure field in the far-field zone, corresponding to the pressure distribution $g(\xi)$ when $\Omega = \Omega_{21}$ (the continuous curve) and the pressure distribution $g^0(\xi) = 0.08333$ ($|\xi| \leq a$) (the dashed curve). It can be seen that in the first of these cases the oscillations in the liquid are localized along the vertical axis and have the form of a highly directional beam.

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